Simultaneous Linear Equations

Simultaneous Equations of the form :→

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Where P, Q, R are functions of x, y, z

Example (1) $\dfrac{x\,dx}{y^2 z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$

Solution:→

Taking first two ratio we write

$$\frac{x\,dx}{y^2 z} = \frac{dy}{xz} \Rightarrow \frac{x\,dx}{y^2} = \frac{dy}{x}$$

$$\Rightarrow x^2 dx = y^2 dy$$

on integration,

$$x^3 - y^3 = c_1 \quad \text{——— (1)}$$

Now taking the first and the third ratios, we get

$$\frac{x\,dx}{y^2 z} = \frac{dz}{y^2}$$

$$\Rightarrow \frac{x\,dx}{z} = dz$$

$$\Rightarrow x\,dx = z\,dz$$

on integration, $x^2 - z^2 = c_2$ is another relation.

The two integrals $x^3 - y^3$ & $x^2 - z^2$ are independent, the required general solution is given by

$$x^3 - y^3 = c_1 \ \& \ x^2 - z^2 = c_2$$

Example ② $\dfrac{dx}{xz(z^2+xy)} = \dfrac{dy}{-yz(z^2+xy)} = \dfrac{dz}{x^4}$

Solution :→
From first two fractions.

$$\frac{dx}{x} = -\frac{dy}{y}$$

$$\Rightarrow xy = c_1$$

from first and third fractions, we get

$$\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4}$$

$$\Rightarrow x^3\,dx = (z^3+c_1 z)\,dz$$

$$\Rightarrow \frac{x^4}{4} - \frac{z^4}{4} - \frac{c_1 z^2}{2} = \text{const.}$$

$$\Rightarrow x^4 - z^4 = 2c_1 z^2 + c_2$$

$$\Rightarrow x^4 - z^4 - 2xyz^2 = c_2 \quad (\text{using } xy = c_1)$$

This is the required general solution.

Example ③ Solve

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)} \quad \text{— (I)}$$

Solution :→ Taking first two fractions (cancelling $x-y$)

$$\frac{dx}{y^2} = -\frac{dy}{x^2}$$

$$\Rightarrow x^2\,dx + y^2\,dy = 0$$

$$\Rightarrow x^3 + y^3 = c_1 \quad \text{— (2)}$$

Now choosing $1, -1, 0$ as multipliers, each fraction of ①

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{y^2(x-y)+x^2(x-y)}$$

$$= \frac{dx-dy}{(x-y)(x^2+y^2)}$$

Now taking third fraction of ① with this fraction we write

$$\frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

$$\Rightarrow \frac{dz}{z} = \frac{d(x-y)}{x-y}$$

$$\Rightarrow \log z - \log(x-y) = \text{constant}$$

$$\Rightarrow \frac{z}{x-y} = c_2$$

$$\Rightarrow z = c_2(x-y) \quad \text{③}$$

The required solution is given by the relations ② & ③.

Example ④ :→ $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

Solution:→ Taking first two ratios we get

$$\frac{dx}{y} = \frac{dy}{x}$$

$$\Rightarrow xdx - ydy = 0$$

on integration

$$x^2 - y^2 = c_1 \qquad \text{——— (1)}$$

Taking the first & the third ratios, we get

$$\frac{dx}{z} = \frac{dz}{x}$$

$$\Rightarrow \quad xdx - zdx = 0$$

on integration,

$$x^2 - z^2 = c_2 \qquad \text{——— (2)}$$

∴ (1) & (2) together give the required solution.

Example ⑤ $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2}$

Example ⑥ $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$

Example ⑦ $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$